Date-31/07/2020 B.Sc (Physics) Part-III, Paper-V, Group-A

Tensor Part-II

In the last lecture note, we have discussed basic idea about tensor, its conventions & notations which we will follow in whole discussion of tensor analysis. We have also discussed Einstein's summation convention dummy index, free index.

We have obtained relation (see earlier note)

$$dx^{i} = \frac{\partial x^{i}}{\partial \bar{x}^{\alpha}} d\bar{x}^{\alpha} \quad \text{— (1)}$$

free index (i), dummy index (α)

$$d\bar{x}^{\alpha} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} dx^{i} \quad \text{— (2)}$$

free index (α), dummy index (i)

Let's discuss an example to clarify more about Einstein's summation convention.

Ex. Let a_i, b_i, c_i, d_i, $1 \le i \le N$, be four sets of N quantities each. Then according to the Einstein's summation convention, we have

$$a_i b_i \equiv a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_N b_N \quad \text{— (3)}$$

and

$$a_i b_j c_j \equiv a_i b_1 c_1 + a_i b_2 c_2 + a_i b_3 c_3 + \cdots + a_i b_N c_N \quad \text{— (4)}$$

i is free index here ~~~~ have fixed value between 1 to N.

Eq. 3 can also be written as

$$a_i b_i = a_j \cdot b_j = a_k b_k = a_r b_r = a_l b_l \text{, etc} \quad (5)$$

In above equation same index is occuring twice in a term. ~~So~~ These are dummy indices.

Again $$a_i b_j c_j = a_i b_k c_k = a_i b_l c_l \quad (6)$$

~~i is dummy index in above equation~~

j, k, l are dummy indices in above ~~equation~~ expression which cannot be replaced by "i" since 'i' appears in the same term.

Therefore, $$a_i b_j c_j \neq a_i b_i c_i \quad (7)$$

The above Eq. can be verified, ~~to~~ write

$$a_i b_i c_i \equiv a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3 + \cdots + a_N b_N c_N \quad (8)$$

See Eq. 8 and 4 are not same. Thus, Eq. 7 is true.

Again consider expressions $a_i b_i c_i d_i$ and $a_i b_i c_j d_j$.
Now,

$$a_i b_i c_i d_i \equiv a_1 b_1 c_1 d_1 + a_2 b_2 c_2 d_2 + a_3 b_3 c_3 d_3 + \cdots + a_N b_N c_N d_N; \quad (9)$$

$$a_i b_i c_j d_j \equiv \left(\sum_{i=1}^{N} a_i b_i\right)\left(\sum_{j=1}^{N} c_j d_j\right)$$

$$= (a_1 b_1 + a_2 b_2 + \cdots + a_N b_N)(c_1 d_1 + c_2 d_2 + \cdots + c_N d_N) \quad (10)$$

From Eqs. 9 and 10

$$a_i b_i c_i d_i \neq a_i b_i c_j d_j$$

Also we can write $a_i b_i c_j d_j = a_i b_i c_k d_k = a_l b_l c_i d_i$, etc

Since coordinates x^i are independent of each other, therefore

$$\frac{dx^i}{dx^j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \quad \text{——— (11)}$$

We define the Kronecker delta symbol by

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{——— (12)}$$

Now Eq. (11) ~~and (12)~~ can be written as

$$\frac{dx^i}{dx^j} = \delta^i_j. \quad \text{—— (13)}$$

Similarly, the coordinates $\bar{x}^\alpha$ are also independent of each other, so that

$$\frac{d\bar{x}^\alpha}{d\bar{x}^\beta} = \delta^\alpha_\beta$$

If x^i are functions of $\bar{x}^\alpha$, then we can write

$$\frac{dx^i}{dx^j} = \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^j}$$

Using Eq. (11) and (12), we can write.

$$\frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^j} = \delta^i_j \quad \text{——— (14)}$$

Similarly, we obtain

$$\frac{\partial \bar{x}^\alpha}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^\beta} = \delta^\alpha_\beta \quad \text{——— (15)}$$